

Higher moments of primes in short intervals I

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Abstract

In this article, we prove an “equivalence” between two higher even moments of primes in short intervals under Riemann Hypothesis. We also provide numerical evidence in support of these asymptotic formulas.

1 Introduction

Recently, Montgomery and Soundararajan [5] studied the moments

$$M_k(N; h) := \sum_{n=1}^N (\psi(n+h) - \psi(n) - h)^k$$

where k is a positive integer, $\psi(x) = \sum_{n \leq x} \Lambda(n)$ and $\Lambda(n)$ is von Mangoldt lambda function. They proved that, under a strong form of Hardy-Littlewood prime- k tuple conjecture, for small $\epsilon > 0$, there is a $\delta > 0$,

$$M_k(N; h) = \mu_k h^{k/2} \int_1^N \left(\log \frac{x}{h} + B \right)^{k/2} dx + O_k(h^{k/2} N^{1-\epsilon}) \quad (1)$$

uniformly for $(\log N)^{15k^2} \leq h \leq N^{1/k-\delta}$ where $\mu_k = 1 \cdot 3 \cdots (k-1)$ if k is even, and $\mu_k = 0$ if k is odd. Here $B = 1 - C_0 - \log 2\pi$ and C_0 denotes Euler's constant. One further expects that (1) holds uniformly for $N^\delta \leq h \leq N^{1-\delta}$. This implies that, for $0 \leq x \leq N$, the distribution of $\psi(x+h) - \psi(x)$ is approximately normal with mean h and variance $h \log N/h$. It contradicts with the prediction of Cramér's model of variance $h \log N$. In the last section, we will show numerical evidence in support of (1).

Now, $M_k(X; h)$ can be written as

$$\int_1^X (\psi(x+h) - \psi(x) - h)^k dx. \quad (2)$$

We also consider the following moments:

$$\widetilde{M}_k(X; \delta) := \int_1^X (\psi(x+\delta x) - \psi(x) - \delta x)^k dx. \quad (3)$$

Goldston and Montgomery [2] showed that, under Riemann Hypothesis (RH), the stronger form of the Pair Correlation Conjecture as formulated by Montgomery [3] is equivalent to an asymptotic formula for (2) in $X^\epsilon \leq h \leq X^{1-\epsilon}$ or an asymptotic formula for (3) in $X^{-1+\epsilon} \leq \delta \leq X^{-\epsilon}$ when $k = 2$. The author generalized these to include the second main terms in [1] (again only when $k = 2$). So, the main purpose of this paper is to prove the “equivalence” between an asymptotic formula for (2) and an asymptotic formula for (3) in appropriate ranges of h and δ for any positive even integer k . Roughly speaking, we have

Theorem 1.1. *Let k be a positive even integer. Assuming RH, the following are equivalent:*

$$(i) \int_1^X (\psi(x+h) - \psi(x) - h)^k dx \sim \mu_k h^{k/2+1} \int_E^{X/h} (\log \frac{x}{E})^{k/2} dx$$

holds uniformly for $X^\epsilon \leq h \leq X^{1-\epsilon}$.

$$(ii) \int_1^X (\psi(x+\delta x) - \psi(x) - \delta x)^k dx \sim \frac{\mu_k}{\frac{k}{2}+1} X^{k/2+1} \delta^{k/2} \left(\log \frac{1}{E\delta} \right)^{k/2}$$

holds uniformly for $X^{-1+\epsilon} \leq \delta \leq X^{-\epsilon}$.

Here $E = 2\pi e^{C_0-1}$. Our method of proof replaces the brute-force calculations in [1]. We will assume RH throughout this paper and k being a positive even integer unless stated otherwise.

This work is part of the author’s 2002 PhD thesis with some improvements.

2 Some preparations

First of all, $\psi(x) = x + O(x^{1/2} \log^2 x)$ by RH (see [7]). One has the following:

$$\int_1^X (\psi(x+\delta x) - \psi(x) - \delta x)^k dx \ll X^{k/2+1} \log^{2k} X \quad (4)$$

for $0 \leq \delta \leq 1$, and

$$\int_1^X (\psi(x+h) - \psi(x) - h)^k dx \ll X^{k/2+1} \log^{2k} X \quad (5)$$

for $0 \leq h \leq X$. Also, estimating trivially, we have

$$\int_1^X (\psi(x+\delta x) - \psi(x) - \delta x)^k dx \ll \int_1^X (\delta x \log X)^k dx \ll \delta^k X^{k+1} \log^k X \quad (6)$$

for $0 \leq \delta \leq 1$, and

$$\int_1^X (\psi(x+h) - \psi(x) - h)^k dx \ll \int_1^X (h \log X)^k dx \ll h^k X \log^k X \quad (7)$$

for $0 \leq h \leq X$. We also need some lemmas.

Lemma 2.1. *For any differentiable function $f(u)$ and any $0 \leq \eta \leq 1$,*

$$\int_T^{(1+\eta)T} f(u) du = \eta T f(T) + O(\eta^2 T^2 \max_{T \leq t \leq (1+\eta)T} |f'(t)|).$$

Proof: By mean-value theorem, we have

$$\begin{aligned} \int_T^{(1+\eta)T} f(u) du &= \eta T f(T + \xi T) \text{ where } 0 \leq \xi \leq \eta \\ &= \eta T (f(T) + \xi T f'(T + \xi' T)) \text{ where } 0 \leq \xi' \leq \xi, \end{aligned}$$

and the lemma follows.

Lemma 2.2. *For any positive integer k , we have*

$$x^k - y^k = (x - y)P(x, y) + (x - y)^k$$

where $P(x, y)$ is some homogeneous polynomial of degree $k - 1$.

Proof: By Factor Theorem, $z - 1$ divides $z^k - 1 - (z - 1)^k$. So,

$$(z - 1)P(z) = z^k - 1 - (z - 1)^k$$

for some integer polynomial $P(z)$ of degree $k - 1$. Set $z = \frac{x}{y}$ and multiply both sides by y^k , we get the desired result.

Lemma 2.3. *For any positive integer k , and any non-negative real numbers α , a_1, a_2, \dots, a_k , we have*

$$(a_1 + a_2 + \dots + a_k)^\alpha \ll_{\alpha, k} a_1^\alpha + a_2^\alpha + \dots + a_k^\alpha.$$

Here, $\ll_{\alpha, k}$ means that the implicit constant may depend on α and k but not on any a_i 's.

Proof: Without loss of generality, suppose that a_1 is the largest among the a_i 's. Then

$$(a_1 + a_2 + \dots + a_k)^\alpha \leq (ka_1)^\alpha \leq k^\alpha (a_1^\alpha + a_2^\alpha + \dots + a_k^\alpha).$$

3 (i) \Rightarrow (ii)

Throughout this and the next section, we think of k as fixed.

Theorem 3.1. *Assume RH. If, for some small $\epsilon > \epsilon_1 > 0$ (small in terms of k),*

$$\int_1^X (\psi(x+h) - \psi(x) - h)^k dx = \mu_k h^{k/2+1} \int_E^{X/h} \left(\log \frac{x}{E}\right)^{k/2} dx + O_k(h^{k/2} X^{1-\epsilon_1}) \quad (8)$$

holds uniformly for $X^\epsilon \leq h \leq X^{1-\epsilon}$, then

$$\int_1^X (\psi(x+\delta x) - \psi(x) - \delta x)^k dx = \frac{\mu_k}{\frac{k}{2} + 1} X^{k/2+1} \delta^{k/2} \left(\log \frac{1}{E\delta} \right)^{k/2} + O_k(\delta^{k/2} X^{k/2+1-\epsilon_2}) \quad (9)$$

holds uniformly for $X^{-1+2\epsilon+2\epsilon_1} \leq \delta \leq X^{-\epsilon}/2$ with some $\epsilon_2 > 0$.

Proof: Our method is that of Saffari and Vaughan [6] employed in [2] and [1]. Let $f(x, h) = \psi(x+h) - \psi(x) - h$. Let $X^{-1+2\epsilon+\epsilon_1} \leq \Delta \leq X^{-\epsilon}$. Say $\Delta = X^{-\mu}$ for some $\epsilon \leq \mu \leq 1 - 2\epsilon - \epsilon_1$. We want to calculate

$$\int_{V/2}^V \int_0^\Delta (\psi(x+\delta x) - \psi(x) - \delta x)^k d\delta dx. \quad (10)$$

Substituting $h = \delta x$, (10) becomes

$$\begin{aligned} & \int_{\Delta V/2}^{\Delta V} \int_{h/\Delta}^V \frac{f(x, h)^k}{x} dx dh + \int_0^{\Delta V/2} \int_{V/2}^V \frac{f(x, h)^k}{x} dx dh \\ &= \int_{\Delta V/2}^{\Delta V} \int_{h/\Delta}^V + \int_{V^\epsilon}^{\Delta V/2} \int_{V/2}^V + \int_0^{V^\epsilon} \int_{V/2}^V = I_1 + I_2 + I_3. \end{aligned}$$

By integration by parts, we have from (8) that

$$\begin{aligned} & \int_U^V \frac{f(x, h)^k}{x} dx \\ &= \left[\frac{1}{x} \int_1^x f(u, h)^k du \right]_U^V + \int_U^V \left(\int_1^x f(u, h)^k du \right) \frac{1}{x^2} dx \\ &= \mu_k h^{k/2+1} \left[\frac{1}{V} \int_E^{V/h} \left(\log \frac{x}{E} \right)^{k/2} dx - \frac{1}{U} \int_E^{U/h} \left(\log \frac{x}{E} \right)^{k/2} dx \right] \\ & \quad + \mu_k h^{k/2+1} \int_U^V \frac{1}{x^2} \int_E^{x/h} \left(\log \frac{u}{E} \right)^{k/2} du dx + O_k(U^{-\epsilon_1} h^{k/2}) \\ &= T_1 + T_2 + O_k(U^{-\epsilon_1} h^{k/2}) \end{aligned}$$

as long as $V^\epsilon \leq h \leq U^{1-\epsilon}$ with $U \leq V \leq 2U$.

$$\begin{aligned} T_2 &= \mu_k h^{k/2} \int_{U/h}^{V/h} \int_E^y \left(\log \frac{u}{E} \right)^{k/2} du d\left(\frac{-1}{y}\right) \\ &= \mu_k h^{k/2} \left[-\frac{h}{V} \int_E^{V/h} \left(\log \frac{u}{E} \right)^{k/2} du + \frac{h}{U} \int_E^{U/h} \left(\log \frac{u}{E} \right)^{k/2} du \right. \\ & \quad \left. + \int_{U/h}^{V/h} \frac{(\log y/E)^{k/2}}{y} dy \right]. \end{aligned}$$

Therefore,

$$\int_U^V \frac{f(x, h)^k}{x} dx = \frac{\mu_k}{\frac{k}{2} + 1} h^{k/2} \left[\left(\log \frac{V}{Eh} \right)^{k/2+1} - \left(\log \frac{U}{Eh} \right)^{k/2+1} \right] + O_k(U^{-\epsilon_1} h^{k/2})$$

as long as $V^\epsilon \leq h \leq U^{1-\epsilon}$ with $U \leq V \leq 2U$. Thus, for I_1 and I_2 to work, we need

$$V^\epsilon \leq \frac{\Delta V}{2} \leq \left(\frac{V}{2}\right)^{1-\epsilon}, \text{ and } V^\epsilon \leq h \leq \left(\frac{h}{\Delta}\right)^{1-\epsilon} \text{ for } \frac{\Delta V}{2} \leq h \leq \Delta V. \quad (11)$$

Since $\Delta = X^\mu$, for $X^{\mu+\epsilon} \leq V \leq X$, one can check that (11) are satisfied. Therefore,

$$\begin{aligned} I_1 &= \frac{\mu_k}{\frac{k}{2}+1} \left[\int_{\Delta V/2}^{\Delta V} h^{k/2} \left(\log \frac{V}{Eh}\right)^{k/2+1} dh - \int_{\Delta V/2}^{\Delta V} h^{k/2} \left(\log \frac{1}{E\Delta}\right)^{k/2+1} dh \right] \\ &\quad + O_k(\Delta^{k/2+1} V^{k/2+1-\epsilon_1}), \\ I_2 &= \frac{\mu_k}{\frac{k}{2}+1} \left[\int_0^{\Delta V/2} h^{k/2} \left(\log \frac{V}{Eh}\right)^{k/2+1} dh - \int_0^{\Delta V/2} h^{k/2} \left(\log \frac{V/2}{Eh}\right)^{k/2+1} dh \right] \\ &\quad + O_k(V^{(k/2+1)\epsilon} (\log V)^{k/2+1}) + O_k(\Delta^{k/2+1} V^{k/2+1-\epsilon_1}), \\ I_3 &\ll V^\epsilon V^{k\epsilon} \log^k V, \end{aligned}$$

Combining these, (10) equals

$$\begin{aligned} &\frac{\mu_k}{\frac{k}{2}+1} \left[\int_0^{\Delta V} h^{k/2} \left(\log \frac{V}{Eh}\right)^{k/2+1} dh - \int_0^{\Delta V/2} h^{k/2} \left(\log \frac{V/2}{Eh}\right)^{k/2+1} dh \right. \\ &\quad \left. - \left(\log \frac{1}{E\Delta}\right)^{k/2+1} \int_{\Delta V/2}^{\Delta V} h^{k/2} dh \right] + O(V^{(k+1)\epsilon} \log^k V) + O_k(\Delta^{k/2+1} V^{k/2+1-\epsilon_1}) \end{aligned}$$

for $X^{\mu+\epsilon} \leq V \leq X$. Now, replacing V by $X2^{-l}$ in the above, summing over $0 \leq l \leq M = \lceil \frac{(1-\mu-\epsilon)\log X}{\log 2} \rceil$,

$$\begin{aligned} &\int_{X/2^M}^X \int_0^\Delta (\psi(x+\delta x) - \psi(x) - \delta x)^k d\delta dx \\ &= \frac{\mu_k}{\frac{k}{2}+1} \left[\int_0^{\Delta X} h^{k/2} \left(\log \frac{X}{Eh}\right)^{k/2+1} dh - \frac{1}{\frac{k}{2}+1} \left(\log \frac{1}{E\Delta}\right)^{k/2+1} (\Delta X)^{k/2+1} \right] \\ &\quad + O_k(X^{(k+1)\epsilon} \log^k X) + O_k(\Delta^{k/2+1} X^{k/2+1-\epsilon_1}) \\ &= \frac{\mu_k}{\frac{k}{2}+1} \int_0^{\Delta X} h^{k/2} \left(\log \frac{X}{Eh}\right)^{k/2} dh + O_k(\Delta^{k/2+1} X^{k/2+1-\epsilon_1}) \end{aligned} \quad (12)$$

by integration by parts, and as $\Delta = X^{-\mu} \geq X^{-1+2\epsilon+\epsilon_1}$,

$$X^{(k+1)\epsilon} \log^k X \ll X^{(k+2)\epsilon+(k/2)\epsilon_1} \ll \Delta^{k/2+1} X^{k/2+1-\epsilon_1}.$$

Using (6),

$$\int_1^{X/2^M} \int_0^\Delta (\psi(x+\delta x) - \psi(x) - \delta x)^k d\delta dx \ll_k \Delta^{k+1} (X^{\mu+\epsilon})^{k+1} \log^k X. \quad (13)$$

But, since $\mu \leq 1 - 2\epsilon - \epsilon_1$,

$$\begin{aligned}\Delta^{k/2+1} X^{k/2+1-\epsilon_1} &= \Delta^{k+1} X^{k/2+1+(k/2)\mu-\epsilon_1} = \Delta^{k+1} X^{(k+1)\mu+(k/2+1)(1-\mu)-\epsilon_1} \\ &\geq \Delta^{k+1} X^{(k+1)\mu+(k/2+1)(2\epsilon+\epsilon_1)} \gg \Delta^{k+1} (X^{\mu+\epsilon})^{k+1} \log^k X.\end{aligned}\tag{14}$$

Combining (12), (13) and (14), we have

$$\begin{aligned}&\int_0^\Delta \int_1^X (\psi(x+\delta x) - \psi(x) - \delta x)^k dx d\delta \\ &= \frac{\mu_k}{\frac{k}{2}+1} \int_0^{\Delta X} h^{k/2} \left(\log \frac{X}{Eh}\right)^{k/2} dh + O_k(\Delta^{k/2+1} X^{k/2+1-\epsilon_1})\end{aligned}\tag{15}$$

for $X^{-1+2\epsilon+\epsilon_1} \leq \Delta \leq X^{-\epsilon}$.

We now deduce (9) from (15). Set $\eta = X^{-2\epsilon_1/3}$. By Lemma 2.1, one has for $X^{-1+2\epsilon+2\epsilon_1} \leq \Delta \leq X^{-\epsilon}/2$,

$$\begin{aligned}&\int_\Delta^{(1+\eta)\Delta} \int_1^X (\psi(x+\delta x) - \psi(x) - \delta x)^k dx d\delta \\ &= \frac{\mu_k}{\frac{k}{2}+1} \int_{\Delta X}^{(1+\eta)\Delta X} h^{k/2} \left(\log \frac{X}{Eh}\right)^{k/2} dh + O_k(\Delta^{k/2+1} X^{k/2+1-\epsilon_1}) \\ &= \frac{\mu_k}{\frac{k}{2}+1} (\Delta X)^{k/2+1} \left(\log \frac{1}{E\Delta}\right)^{k/2} \eta + O_k\left(\eta^2 (\Delta X)^{k/2+1} \left(\log \frac{1}{\Delta}\right)^{k/2}\right) \\ &\quad + O_k(\Delta^{k/2+1} X^{k/2+1-\epsilon_1}).\end{aligned}\tag{16}$$

Let $g(x, \delta x) = f(x, \Delta x)$ for $\Delta \leq \delta \leq (1+\eta)\Delta$. Then one can easily check that $f(x, \delta x) - g(x, \delta x) = f((1+\Delta)x, (\delta - \Delta)x)$. So,

$$\begin{aligned}&\int_\Delta^{(1+\eta)\Delta} \int_1^X (f(x, \delta x) - g(x, \delta x))^k dx d\delta = \int_0^{\frac{\eta\Delta}{1+\Delta}} \int_{1+\Delta}^{(1+\Delta)X} f(x, \delta x)^k dx d\delta \\ &\ll_k (\eta X \Delta)^{k/2+1} \left(\log \frac{1}{\eta\Delta}\right)^{k/2}\end{aligned}\tag{17}$$

by (15), the choice of η and the range of Δ . Thus, by Lemma 2.3, (16) and (17),

$$\begin{aligned}&\int_\Delta^{(1+\eta)\Delta} \int_1^X g(x, \delta x)^k dx d\delta \ll_k \int \int |f(x, \delta x)|^k + \int \int |f(x, \delta x) - g(x, \delta x)|^k \\ &\ll_k \eta X^{k/2+1} \Delta^{k/2+1} \left(\log \frac{1}{\Delta}\right)^{k/2} + \Delta^{k/2+1} X^{k/2+1-\epsilon_1}.\end{aligned}\tag{18}$$

By Lemma 2.2 and Holder's inequality,

$$\begin{aligned}
& \int_{\Delta}^{(1+\eta)\Delta} \int_1^X f(x, \delta x)^k - g(x, \delta x)^k dx d\delta \\
&= \int_{\Delta}^{(1+\eta)\Delta} \int_1^X P(f, g)(f - g) + \int_{\Delta}^{(1+\eta)\Delta} \int_1^X (f - g)^k \\
&\ll \left(\int \int |P(f, g)|^{k/(k-1)} \right)^{(k-1)/k} \left(\int \int |f - g|^k \right)^{1/k} + \int \int |f - g|^k \\
&= J_1^{(k-1)/k} J_2^{1/k} + J_2
\end{aligned}$$

where $P(x, y)$ is a homogeneous polynomial of degree $k - 1$.

$$J_2 \ll_k \eta^{k/2+1} X^{k/2+1} \Delta^{k/2+1} (\log \frac{1}{\eta\Delta})^{k/2}$$

by (17). And

$$\begin{aligned}
J_1 &\ll_k \int \int \left(\sum_{i+j=k-1} |f|^i |g|^j \right)^{k/(k-1)} \\
&\ll_k \sum_{i+j=k-1} \int \int (|f| + |g|)^k \text{ by binomial theorem} \\
&\ll_k \int \int f^k + g^k \text{ by Lemma 2.3} \\
&\ll_k \eta X^{k/2+1} \Delta^{k/2+1} (\log \frac{1}{\eta\Delta})^{k/2} + \Delta^{k/2+1} X^{k/2+1-\epsilon_1} \text{ by (16) and (18)} \\
&\ll_k \eta X^{k/2+1} \Delta^{k/2+1} (\log \frac{1}{\eta\Delta})^{k/2} \text{ as } \eta = X^{-2\epsilon_1/3}.
\end{aligned}$$

Consequently, by Lemma 2.3,

$$\int \int f^k - g^k \ll_k X^{k/2+1} \Delta^{k/2+1} \eta^{3/2} (\log \frac{1}{\eta\Delta})^{k/2}. \quad (19)$$

Therefore, by (19) and (16),

$$\begin{aligned}
& \eta\Delta \int_1^X (\psi(x + \Delta x) - \psi(x) - \Delta x)^k dx = \int \int g^k \\
&= \int \int f^k + O\left(X^{k/2+1} \Delta^{k/2+1} \eta^{3/2} (\log \frac{1}{\eta\Delta})^{k/2}\right) \\
&= \frac{\mu_k}{\frac{k}{2} + 1} (X\Delta)^{k/2+1} \left(\log \frac{1}{E\Delta}\right)^{k/2} \eta + O_k(\Delta^{k/2+1} X^{k/2+1-\epsilon_1}) \\
&\quad + O_k\left(X^{k/2+1} \Delta^{k/2+1} \eta^{3/2} (\log \frac{1}{\eta\Delta})^{k/2}\right).
\end{aligned}$$

Dividing through by $\eta\Delta$, we have

$$\begin{aligned} & \int_1^X (\psi(x + \Delta x) - \psi(x) - \Delta x)^k dx \\ &= \frac{\mu_k}{\frac{k}{2} + 1} X^{k/2+1} \Delta^{k/2} \left(\log \frac{1}{E\Delta} \right)^{k/2} + O_k \left(\frac{\Delta^{k/2} X^{k/2+1-\epsilon_1}}{\eta} \right) \\ & \quad + O_k \left(X^{k/2+1} \Delta^{k/2} \eta^{1/2} \left(\log \frac{1}{\eta\Delta} \right)^{k/2} \right). \end{aligned}$$

Finally, recall $\eta = X^{-2\epsilon_1/3}$, one has the error terms $\ll X^{k/2+1-\epsilon_1/4} \Delta^{k/2}$. So, the theorem is true with $\epsilon_2 = \frac{\epsilon_1}{4}$.

4 (ii) \Rightarrow (i)

Theorem 4.1. Assume RH. If, for some small $\epsilon > \epsilon_1 > 0$ (small in terms of k),

$$\int_1^X (\psi(x + \delta x) - \psi(x) - \delta x)^k dx = \frac{\mu_k}{\frac{k}{2} + 1} X^{k/2+1} \delta^{k/2} \left(\log \frac{1}{E\delta} \right)^{k/2} + O_k(\delta^{k/2} X^{k/2+1-\epsilon_1}) \quad (20)$$

holds uniformly for $X^{-1+\epsilon} \leq \delta \leq X^{-\epsilon}$, then

$$\int_1^X (\psi(x + h) - \psi(x) - h)^k dx = \mu_k h^{k/2+1} \int_E^{X/h} \left(\log \frac{x}{E} \right)^{k/2} dx + O_k(h^{k/2} X^{1-\epsilon_2}) \quad (21)$$

holds uniformly for $X^{2\epsilon+\epsilon_1} \leq h \leq X^{1-(k/2+1)\epsilon-2\epsilon_1}/2$ with some $\epsilon_2 > 0$.

Proof: Let $f(x, h) = \psi(x + h) - \psi(x) - h$. Let $X^{2\epsilon} \leq H \leq X^{1-(k/2+1)\epsilon-2\epsilon_1}$. Say $H = X^\mu$ for some $2\epsilon \leq \mu \leq 1 - (k/2 + 1)\epsilon - 2\epsilon_1$. First, we calculate

$$\int_{V/2}^V \int_0^H (\psi(x + h) - \psi(x) - h)^k dh dx. \quad (22)$$

Substituting $\delta = \frac{h}{x}$, (22) becomes

$$\begin{aligned} & \int_{H/V}^{2H/V} \int_{V/2}^{H/\delta} f(x, \delta x)^k x dx d\delta + \int_0^{H/V} \int_{V/2}^V f(x, \delta x)^k x dx d\delta \\ &= \int_{H/V}^{2H/V} \int_{V/2}^{H/\delta} + \int_{(\frac{V}{2})^{-1+\epsilon}}^{H/V} \int_{V/2}^V + \int_0^{(\frac{V}{2})^{-1+\epsilon}} \int_{V/2}^V = I_1 + I_2 + I_3. \end{aligned}$$

By integration by parts, we have from (20) that

$$\begin{aligned} & \int_U^V f(x, \delta x)^k x dx \\ &= \frac{\mu_k}{\frac{k}{2} + 2} \delta^{k/2} \left(\log \frac{1}{E\delta} \right)^{k/2} \left[V^{k/2+2} - U^{k/2+2} \right] + O_k(\delta^{k/2} V^{k/2+2-\epsilon_1}) \end{aligned}$$

as long as $U^{-1+\epsilon} \leq \delta \leq V^{-\epsilon}$ with $U \leq V \leq 2U$. In order for this to work for I_1 and I_2 , we need

$$\left(\frac{V}{2}\right)^{-1+\epsilon} \leq \frac{H}{V} \leq V^{-\epsilon}, \text{ and } \left(\frac{V}{2}\right)^{-1+\epsilon} \leq \delta \leq \left(\frac{H}{\delta}\right)^{-\epsilon} \text{ for } \frac{H}{V} \leq \delta \leq \frac{2H}{V}. \quad (23)$$

Since $H = X^\mu$, one can check that (23) are satisfied for $X^{\mu+\epsilon} \leq V \leq X$. Thus,

$$\begin{aligned} I_1 &= \frac{\mu_k}{\frac{k}{2}+2} \int_{H/V}^{2H/V} \left[\left(\frac{H}{\delta}\right)^{k/2+2} - \left(\frac{V}{2}\right)^{k/2+2} \right] \delta^{k/2} \left(\log \frac{1}{E\delta}\right)^{k/2} d\delta \\ &\quad + O_k(H^{k/2+1}V^{1-\epsilon_1}), \\ I_2 &= \frac{\mu_k}{\frac{k}{2}+2} \left[V^{k/2+2} - \left(\frac{V}{2}\right)^{k/2+2} \right] \int_0^{H/V} \delta^{k/2} \left(\log \frac{1}{E\delta}\right)^{k/2} d\delta \\ &\quad + O_k(V^{1+(k/2+1)\epsilon}(\log V)^{k/2}) + O_k(H^{k/2+1}V^{1-\epsilon_1}), \\ I_3 &\ll V^{-1+\epsilon}V^2V^{k\epsilon} \log^k V = V^{1+(k+1)\epsilon} \log^k V. \end{aligned}$$

Let $\nu_k = \mu_k/(\frac{k}{2}+2)$, then (22)

$$\begin{aligned} &= \nu_k H^{k/2+2} \int_{H/V}^{2H/V} \frac{1}{\delta^2} \left(\log \frac{1}{E\delta}\right)^{k/2} d\delta + \nu_k V^{k/2+2} \int_0^{H/V} \delta^{k/2} \left(\log \frac{1}{E\delta}\right)^{k/2} d\delta \\ &\quad - \nu_k \left(\frac{V}{2}\right)^{k/2+2} \int_0^{2H/V} \delta^{k/2} \left(\log \frac{1}{E\delta}\right)^{k/2} d\delta \\ &\quad + O_k(V^{1+(k+1)\epsilon} \log^k V) + O_k(H^{k/2+1}V^{1-\epsilon_1}) \end{aligned}$$

when $X^{\mu+\epsilon} \leq V \leq X$. Now, replacing V by $X2^{-l}$ in the above, summing over $0 \leq l \leq M = \lceil \frac{(1-\mu-\epsilon)\log X}{\log 2} \rceil$,

$$\begin{aligned} &\int_{X/2^M}^X \int_0^H (\psi(x+h) - \psi(x) - h)^k dh dx \\ &= \nu_k H^{k/2+2} \int_{H/X}^{2^{M+1}H/X} \frac{1}{\delta^2} \left(\log \frac{1}{E\delta}\right)^{k/2} d\delta + \nu_k X^{k/2+2} \int_0^{H/X} \delta^{k/2} \left(\log \frac{1}{E\delta}\right)^{k/2} d\delta \\ &\quad - \nu_k \left(\frac{X}{2^{M+1}}\right)^{k/2+2} \int_0^{2^{M+1}H/X} \delta^{k/2} \left(\log \frac{1}{E\delta}\right)^{k/2} d\delta + O_k(H^{k/2+1}X^{1-\epsilon_1}) \\ &= \nu_k H^{k/2+2} \int_{H/X}^{1/E} \frac{1}{\delta^2} \left(\log \frac{1}{E\delta}\right)^{k/2} d\delta + \nu_k X^{k/2+2} \int_0^{H/X} \delta^{k/2} \left(\log \frac{1}{E\delta}\right)^{k/2} d\delta \\ &\quad + O_k(H^{k/2+1}X^{1-\epsilon_1}). \end{aligned} \quad (24)$$

because, as $X^{2\epsilon} \leq H$, $X^{1+(k+1)\epsilon} \log^k X \ll H^{k/2+1}X^{1-\epsilon_1}$. Also, the terms involving 2^{M+1} are absorbed into the error term as $\mu \leq 1 - \epsilon - 2\epsilon_1$. Using (5),

$$\int_1^{X/2^M} \int_0^H (\psi(x+h) - \psi(x) - h)^k dh dx \ll H(X^{\mu+\epsilon})^{k/2+1} \log^{2k} X. \quad (25)$$

But, since $H = X^\mu \leq X^{1-(k/2+1)\epsilon-2\epsilon_1}$,

$$\begin{aligned} H(X^{\mu+\epsilon})^{k/2+1} \log^{2k} X &\leq X^{1-(k/2+1)\epsilon-2\epsilon_1} H^{k/2+1} X^{(k/2+1)\epsilon} \log^{2k} X \\ &\ll H^{k/2+1} X^{1-\epsilon_1}. \end{aligned} \quad (26)$$

Combining (24), (25) and (26), we have

$$\begin{aligned} &\int_0^H \int_1^X (\psi(x+h) - \psi(x) - h)^k dx dh \\ &= \nu_k H^{k/2+2} \int_{H/X}^{1/E} \frac{1}{\delta^2} \left(\log \frac{1}{E\delta} \right)^{k/2} d\delta + \nu_k X^{k/2+2} \int_0^{H/X} \delta^{k/2} \left(\log \frac{1}{E\delta} \right)^{k/2} d\delta \\ &\quad + O_k(H^{k/2+1} X^{1-\epsilon_1}) \end{aligned} \quad (27)$$

for $X^{2\epsilon} \leq H \leq X^{1-(k/2+1)\epsilon-2\epsilon_1}$.

We now deduce (21) from (27). Set $\eta = X^{-2\epsilon_1/3}$. For $X^{2\epsilon+\epsilon_1} \leq H \leq X^{1-(k/2+1)\epsilon-2\epsilon_1}/2$,

$$\begin{aligned} &\int_H^{(1+\eta)H} \int_1^X (\psi(x+h) - \psi(x) - h)^k dx dh \\ &= \nu_k ((1+\eta)H)^{k/2+2} \int_{(1+\eta)H/X}^{1/E} \frac{1}{\delta^2} \left(\log \frac{1}{E\delta} \right)^{k/2} d\delta \\ &\quad - \nu_k H^{k/2+2} \int_{H/X}^{1/E} \frac{1}{\delta^2} \left(\log \frac{1}{E\delta} \right)^{k/2} d\delta \\ &\quad + \nu_k X^{k/2+2} \int_{H/X}^{(1+\eta)H/X} \delta^{k/2} \left(\log \frac{1}{E\delta} \right)^{k/2} d\delta + O_k(H^{k/2+1} X^{1-\epsilon_1}) \\ &= -\nu_k H^{k/2+2} \int_{H/X}^{(1+\eta)H/X} \frac{1}{\delta^2} \left(\log \frac{1}{E\delta} \right)^{k/2} d\delta + \mu_k \eta H^{k/2+2} \int_{H/X}^{1/E} \frac{1}{\delta^2} \left(\log \frac{1}{E\delta} \right)^{k/2} d\delta \\ &\quad + \nu_k X^{k/2+2} \int_{H/X}^{(1+\eta)H/X} \delta^{k/2} \left(\log \frac{1}{E\delta} \right)^{k/2} d\delta \\ &\quad + O_k\left(\eta^2 H^{k/2+1} X \left(\log \frac{X}{H} \right)^{k/2}\right) + O_k(H^{k/2+1} X^{1-\epsilon_1}) \\ &= \mu_k \eta H^{k/2+2} \int_{H/X}^{1/E} \frac{1}{\delta^2} \left(\log \frac{1}{E\delta} \right)^{k/2} d\delta \\ &\quad + O_k\left(\eta^2 H^{k/2+1} X \left(\log \frac{X}{H} \right)^{k/2}\right) + O_k(H^{k/2+1} X^{1-\epsilon_1}) \end{aligned}$$

by Lemma 2.1. Therefore

$$\begin{aligned}
& \int_H^{(1+\eta)H} \int_1^X (\psi(x+h) - \psi(x) - h)^k dx dh \\
&= \mu_k \eta H^{k/2+2} \int_E^{X/H} \left(\log \frac{u}{E}\right)^{k/2} du \\
&+ O_k\left(\eta^2 H^{k/2+1} X \left(\log \frac{X}{H}\right)^{k/2}\right) + O_k(H^{k/2+1} X^{1-\epsilon_1}).
\end{aligned} \tag{28}$$

Let $g(x, h) = f(x, H)$ for $H \leq h \leq (1 + \eta)H$. Again, one can check that $f(x, h) - g(x, h) = f(x + H, h - H)$. So,

$$\begin{aligned}
\int_H^{(1+\eta)H} \int_1^X (f(x, h) - g(x, h))^k dx dh &= \int_0^{\eta H} \int_{1+H}^{X+H} f(x, h)^k dx dh \\
&\ll_k \eta^{k/2+1} X H^{k/2+1} \left(\log \frac{X}{\eta H}\right)^{k/2}
\end{aligned} \tag{29}$$

by (27) as well as the choice of η and the range of H . Thus, by Lemma 2.3, (28) and (29),

$$\begin{aligned}
\int_H^{(1+\eta)H} \int_1^X g(x, h)^k dx dh &\ll_k \int \int |f(x, h)|^k + \int \int |f(x, h) - g(x, h)|^k \\
&\ll_k \eta X H^{k/2+1} \left(\log \frac{X}{\eta H}\right)^{k/2} + H^{k/2+1} X^{1-\epsilon_1}.
\end{aligned} \tag{30}$$

By Lemma 2.2 and Holder's inequality,

$$\begin{aligned}
& \int_H^{(1+\eta)H} \int_1^X f(x, h)^k - g(x, h)^k dx dh \\
&= \int_H^{(1+\eta)H} \int_1^X P(f, g)(f - g) + \int_H^{(1+\eta)H} \int_1^X (f - g)^k \\
&\ll \left(\int \int |P(f, g)|^{k/(k-1)}\right)^{(k-1)/k} \left(\int \int |f - g|^k\right)^{1/k} + \int \int |f - g|^k \\
&= K_1^{(k-1)/k} K_2^{1/k} + K_2
\end{aligned} \tag{31}$$

where $P(x, y)$ is a homogeneous polynomial of degree $k - 1$. From (29),

$$K_2 \ll_k \eta^{k/2+1} X H^{k/2+1} \left(\log \frac{X}{\eta H}\right)^{k/2}. \tag{32}$$

And similar to the proof in Theorem 3.1,

$$K_1 \ll_k \int \int f^k + g^k \ll_k \eta X H^{k/2+1} \left(\log \frac{X}{\eta H}\right)^{k/2} \tag{33}$$

by (28) and (30). Consequently, by (31), (32), (33) and Lemma 2.3,

$$\int \int f^k - g^k \ll_k \eta^{3/2} X H^{k/2+1} \left(\log \frac{X}{\eta H} \right)^{k/2}. \quad (34)$$

Therefore, by (34) and (28),

$$\begin{aligned} & \eta H \int_1^X (\psi(x+h) - \psi(x) - h)^k dx = \int \int g^k \\ &= \int \int f^k + O_k \left(X H^{k/2+1} \eta^{3/2} \left(\log \frac{X}{\eta H} \right)^{k/2} \right) \\ &= \mu_k \eta H^{k/2+2} \int_E^{X/H} \left(\log \frac{u}{E} \right)^{k/2} du + O_k (H^{k/2+1} X^{1-\epsilon_1}) \\ & \quad + O_k \left(X H^{k/2+1} \eta^{3/2} \left(\log \frac{X}{\eta H} \right)^{k/2} \right). \end{aligned}$$

Divide through by ηH and recall $\eta = X^{-2\epsilon_1/3}$, we get the theorem with $\epsilon_2 = \frac{\epsilon_1}{4}$.

5 Numerical evidence

In Montgomery and Soundararajan [4], they got some numerical data for the actual values of $M_k(X; h)$. One has the following table:

For $X = 10^{10}$ and $h = 10^5$.

k	Actual value of $M_k(X; h)$	Result from formula (i) of Theorem 1.1
2	$9.0663 * 10^{15}$	$9.0978 * 10^{15}$
4	$2.4995 * 10^{22}$	$2.5131 * 10^{22}$
6	$1.1573 * 10^{29}$	$1.1675 * 10^{29}$

Using a C program, we get some numerical evidence in support of the truth of (ii) in Theorem 1.1.

For $X = 10^8$ and $\delta = 10^{-4}$:

k	Actual value of $\widetilde{M}_k(X; \delta)$	Result from formula (ii) of Theorem 1.1
2	$4.0075 * 10^{12}$	$3.8976 * 10^{12}$
4	$6.5161 * 10^{17}$	$6.0766 * 10^{17}$
6	$1.9592 * 10^{23}$	$1.7763 * 10^{23}$

For $X = 10^{10}$ and $\delta = 10^{-5}$:

k	Actual value of $\widetilde{M}_k(X; \delta)$	Result from formula (ii) of Theorem 1.1
2	$5.0527 * 10^{15}$	$5.0485 * 10^{15}$
4	$1.0210 * 10^{22}$	$1.0195 * 10^{22}$
6	$3.8645 * 10^{28}$	$3.8602 * 10^{28}$

References

- [1] T.H. Chan, *More Precise Correlation of Zeros and Primes in Short Intervals*, J. London Math. Soc. (2) **68**, 2003, no. 3, 579 - 598.
- [2] D.A. Goldston and H.L. Montgomery, *On pair correlations of zeros and primes in short intervals*, Analytic Number Theory and Diophantine Problems (Stillwater, OK, July 1984), Prog. Math. **70**, Birkhauser, Boston, 1987, pp. 183-203.
- [3] H.L. Montgomery, *The pair correlation of zeros of the zeta function*, Analytic Number Theory (St. Louis Univ., 1972), Proc. Sympos. Pure Math. **24**, Amer. Math. Soc., Providence, 1973, pp. 181-193.
- [4] H.L. Montgomery and K. Soundararajan, *Beyond pair correlation*, Paul Erdős and his mathematics, I (Budapest, 1999), Bolyai Soc. Math. Stud. **11**, János Bolyai Math. Soc., Budapest, 2002, pp. 507-514.
- [5] H.L. Montgomery and K. Soundararajan, *Primes in Short Intervals*, preprint.
- [6] B. Saffari and R.C. Vaughan, *On the fractional parts of x/n and related sequences II*, Ann. Inst. Fourier (Grenoble) (2) **27**, 1977, 1-30.
- [7] H. von Koch, *Sur la distribution des nombres premiers*, Acta Math. **24**, 1901, 159-182.

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